

Theorem of Levinson via the Spectral Density

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We deduce Levinson's theorem in non-relativistic quantum mechanics in one dimension as a sum rule for the spectral density constructed from asymptotic data. We assume a self-adjoint Hamiltonian which guarantees completeness; the potential needs not to be isotropic and a zero-energy resonance is automatically taken into account. Peculiarities of this one-dimension case are explained because of the "critical" character of the free case $u(x) = 0$, in the sense that any attractive potential forms at least a bound state. We believe this method is more general and direct than the usual one in which one proves the theorem first for single wave modes and performs analytical continuation.

KEY WORDS: non-relativistic quantum mechanics; Levinson theorem; spectral density.

1. INTRODUCTION

Two generic results in potential scattering stand on their own, and hold with wide generality. The first, the *optical theorem* (Feinberg, 1932) stems from the fact that the scattered "matter" is taken away from the incoming wave, and hence the scattering center "casts a shadow" in the forward direction as to produce negative interference with the incoming beam; therefore a relation must exist between the total scattering cross section and the forward scattering amplitude. Originally proven in partial waves for 3D scattering, the theorem holds with much more generality; a simple but very general proof is offered in Boya and Murray (1994). The theorem can be seen as a result of the "completeness relation" in ordinary space (also called orthogonality), at a given energy.

The second result is *Levinson's theorem*, which in a way can be seen also as a consequence of completeness in *momentum space*. In its primitive form of 1949 Levinson's theorem reads (Levinson, 1949a)

$$n_\ell = (1/\pi)(\delta_\ell(0) - \delta_\ell(\infty)) \quad (1)$$

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for the number n_ℓ of bound states of angular momentum ℓ in a generic central potential $u(\vec{r}) = u(|\vec{r}|)$ which produces a phase shift $\delta_\ell(k)$ for scattering with energy $E = k^2$; the proof came up as a byproduct of studies on uniqueness of potentials with a given phase shift (Levinson, 1949b). Formula (1) was rediscovered in de Witt (1956). In (Jauch, 1957) it is established the theorem as a consequence of the completeness relation for scattering states and set it up in the general frame of operator theory.

The philosophy was that, if a potential generates bound states, there should be a relation between them and scattering states as the completeness relation has to be “shared” among them. Indeed, Levinson’s theorem is the only relation between bound states and scattering states as it can be deduced from inverse scattering theory. Since the work of Jauch, many studies followed and we mention in particular the elementary deduction by Wellner (1964) for s -waves in $3D$, which we shall generalize in this paper and the later studies of Newton relating the theorem to the inverse problem in 1, 2 or 3 dimensions (Newton, 1977, 1983, 1984, 1982).

It is considerably more difficult to prove Levinson’s theorem than the optical theorem, although both share complementary physical foundations. In this paper we prove the theorem as a sum rule for the spectral density, which we take as more fundamental entity; we shall work in one dimension with local potential where all the features of the problem already show up. Indications for $D > 1$ will be given at the end of the paper.

In our one-dimensional problem we shall *not* assume parity invariance (corresponding to central forces in $D = 3$) nor we shall exclude a zero-energy resonance. We were motivated by the spectral density considerations of Niemi and Semenoff (1986) for fermions in solitonic backgrounds whereas the Wronskian-like technique is adopted from Wellner as stated. The plan of the paper is as follows: in Section 2 we set up the scattering problem in $D = 1$, mainly to motivate notation for direct (left to right) and right to left (“zurdo”) scattering, unitarity of the S -matrix, etc. In Section 3 we introduce the spectral density which needs to be regularized in a box, but in the definition of *relative* spectral density the space cut-off can (and will) be removed. As stated, Levinson’s theorem will appear clearly as a sum rule for the relative spectral density.

In Section 4 we shall handle two simple examples which can be worked out directly, namely the solitonic Pöschl-Teller potential $u(x) = -2 \operatorname{sech}^2 x$ and the delta potential $u(x) = g \delta(x)$; the difference of generic vs. critical potential will be cleared up, as well as the one-half factor already noted by Barton (1985), and present in the one-dimensional case (another factor should appear in two dimensions) (Cheney, 1984). The next section will exhibit our general treatment of the relative spectral density for an arbitrary local potential $u = u(x)$ (of course, decaying at $x \rightarrow \pm\infty$ fast enough to allow for scattering). We shall emphasize that the spectral density is a “hard datum” (i.e. spectrum-dependent as opposed

to the rest of potential parameters or “soft data”); the spectral density is given in terms of the derivative of the phase of the forward amplitude, this amplitude *itself* being a hard datum as well, explaining a result used routinely in the KdV-like evolution equations. In Section 6 we carry out the momentum integration of the density to produce the general form of Levinson’s theorem; we comment briefly on the relation with the determinant of the S -matrix which suggests an interpretation of the theorem as an index theorem for the bound states. In Section 7 we set up the procedure for an *arbitrary* dimension, thus generalizing the method of Wellner; noncentral and critical potentials (i.e., producing a zero-energy resonance) are easily included; finally we make some comments on non-local potentials and add some concluding remarks.

2. SCATTERING IN ONE-DIMENSION

Let

$$\psi''(x) + k^2\psi(x) = u(x)\psi(x) \quad (2)$$

be the Schrödinger equation in one dimension for a local (hence real because hermitian) potential allowing scattering, i.e. satisfying (Deift and Trubowitz, 1979)

$$\int_{-\infty}^{\infty} (1+x^2)|u(x)| dx < \infty \quad (3)$$

for positive $E = k^2$. The $D = 1$ scattering has card $S^o =$ two modes, *direct* (incoming wave towards the right), with the asymptotic solution:

$$\psi(x) \longrightarrow \begin{cases} \exp(ikx) + b(k)\exp(-ikx) & \text{for } x \ll 0 \\ t(k)\exp(ikx) & \text{for } x \gg 0 \end{cases} \quad (4)$$

and *zurdo* scattering: the incoming wave travels towards the left

$$\tilde{\psi}(x) \longrightarrow \begin{cases} \exp(-ikx) + \tilde{b}(k)\exp(ikx) & \text{for } x \gg 0 \\ \tilde{t}(k)\exp(-ikx) & \text{for } x \ll 0. \end{cases} \quad (5)$$

The amplitudes $f(k) = t(k) - 1$ and $b(k)$ give the scattering coefficients by the relations

$$\sigma_{\rightarrow} = |f(k)|^2, \quad \sigma_{\leftarrow} = |b(k)|^2 \quad (6)$$

together with

$$|t(k)|^2 + |b(k)|^2 = 1 \quad \text{or} \quad |f(k)|^2 + |b(k)|^2 \equiv \sigma_{\text{tot}} = -2\Re f(k), \quad (7)$$

the *optical theorem* in one dimension. The S -matrix has two channels only

$$S(k) = \begin{pmatrix} t(k) & \tilde{b}(k) \\ b(k) & \tilde{t}(k) \end{pmatrix}. \tag{8}$$

Unitarity $S^\dagger S = SS^\dagger$ leads to the important relations (with $a = |a| \exp(i\Phi_a)$ for any amplitude)

$$|t(k)|^2 + |b(k)|^2 = |t(k)|^2 + |\tilde{b}(k)|^2 = |\tilde{t}(k)|^2 + |\tilde{b}(k)|^2 = 1 \tag{9}$$

and

$$t(k)b(k)^* + \tilde{b}(k)\tilde{t}(k)^* = 0, \quad \Phi[t(k)] + \Phi[\tilde{t}(k)] = \pi + \Phi[b(k)] + \Phi[\tilde{b}(k)]. \tag{10}$$

Now the potential being local (and hermitian of course) is real, so *time reversal* holds; it follows that

$$t(k) = \tilde{t}(k) \tag{11}$$

whereas an even potential would imply similarly $b(k) = \tilde{b}(k)$, which we do *not* assume. To see (11), notice $t(k)$ is defined as the *transition* from \vec{k}_{in} to \vec{k}_{out} . Time reversal changes \vec{k} to $-\vec{k}$ and *in* to *out*; hence $t(k)$ goes to $\tilde{t}(k)$. Also we shall take $k \geq 0$ for direct scattering and $k \leq 0$ for *zurdo* scattering; indeed then $\tilde{t}(k) = -t(-k)$ for the above reasons.

We remind that we also use the terms *hard data* for spectral data, namely the spectral density, and *soft data* for orientation data, to wit, normalization constants for bound states and the phase of $b(k)$.

3. THE SPECTRAL DENSITY

We recall first elementary properties of matrices. *Completeness* for a *diagonalizable* finite matrix M means

$$M = \sum_m m P_m \quad \text{or} \quad 1| = \sum_m P_m \tag{12}$$

for eigenvalues m and projectors P_m ; the second relation is called *resolution of the identity*. In his work on Quantum Mechanics, von Neumann (1932) extended the classical work of Hilbert on integral equations for hermitian *unbounded* operators with *continuous spectrum*: he called *hypermaximal* (today “self-adjoint”) those hermitian operators H which still support a *resolution of the identity*. The resolution reads

$$1|_{\mathcal{H}} = \sum_{j=1}^N P_j + \int_0^\infty dP(\mu) \tag{13}$$

where \mathcal{H} is the Hilbert space of states, $1| = 1|_{\mathcal{H}}$ is the unit operator, P_j projects to the *finite or infinite* number ($N = 0, 1, \dots$) of *bound* states and the continuum integral, supposed by simplicity extended from 0 to ∞ , as in a standard potential problem, means *projection-valued measures*. Notice the unit operator $1|_{\mathcal{H}}$ is bounded but *not* in the trace class.

The simplest approach to the Levinson's theorem is to state the same resolution for the free system H_o

$$1|_{\mathcal{H}} = \int_0^{\infty} dP^{(o)}(\mu). \quad (14)$$

Subtracting in (13) and then taking traces we get, supposing short-range potentials which support at most *finite* number N of bound states

$$-N = \int_0^{\infty} \text{Tr} d[P(\mu) - P^{(o)}(\mu)] \quad (15)$$

which is, really, the most general (but rather useless) form of the theorem. The idea is now to trade the projectors for scattering amplitudes (or phase shifts). We know of course the *normalized* free continuum wave functions

$$\psi_k^{(o)}(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx). \quad (16)$$

Let us normalize the continuum wave functions to 1, instead to $\sqrt{2\pi}$. Then (15) for our case really means

$$-2\pi N = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx [|\psi_k(x)|^2 - 1] \quad (17)$$

as $dP/dk \sim |k\rangle\langle k|$, taking the traces in x -spaces, defining $\psi_k(x) := \langle x|k\rangle$ and using the double degeneracy $E = (\pm k)^2$ to extend the integral to $\pm\infty$. Expression (17) will be our fundamental formula; the k -integration will be clarified below.

The main concern is to express (17) in terms of the asymptotic data (phase shifts). Define the *relative* spectral density for the problem as

$$\rho_{\text{rel}}(k) := \int_{-\infty}^{\infty} dx [|\psi_k(x)|^2 - 1]. \quad (18)$$

Notice the individual spectral densities diverge, i.e. $\rho_{\text{free}}(k) = \int 1 dx = \infty$; only *relative densities* make sense. The idea of the proof is to relate the spectral densities to scattering data; as we know that the integral of the spectral density gives Levinson's, which express the number N of bound states in terms of the range of the phase shift ($0 \rightarrow \infty$), (see (1)), we expect the spectral densities to be given in terms of derivatives of the phase shifts; we shall see that this is so. Notice also the relative spectral density is a *measure*, that is, something under

an integral symbol; hence some apparent pathologies like delta-type behavior or box-normalization and removal are perfectly legitimate, and not only “heuristic” as some authors state.

4. TWO SIMPLE EXAMPLES: PÖSCHL-TELLER SOLITON AND DELTA POTENTIALS

As a warming-up exercise, let us compute the spectral densities in two simple cases in which the exact continuum wave functions are known. When considering the Pöschl-Teller potential (the standard solitonic potential) given by

$$u(x) = -\frac{2}{\cosh^2 x}, \quad (19)$$

there is a *single bona fide* bound state with energy $E = -1$ and continuum states with *no* reflection since the potential is transparent. Indeed, the whole $E > 0$ wave function $\psi_k(x)$ is obtained from the $u(x) = 0$ case by Darboux’s method (Blecuca *et al.*, 2003). If $D \equiv d/dx$

$$\psi_k(x) \sim (D - \tanh x) \exp(ikx). \quad (20)$$

With the correct normalization included (so $\psi_k(x \ll 0) \sim \exp(ikx)$),

$$\psi_k(x) = \frac{(ik - \tanh x)}{ik + 1} \exp(ikx). \quad (21)$$

The relative spectral density is therefore

$$\rho_{\text{rel}}(k) = \int_{-\infty}^{\infty} [|\psi_k(x)|^2 - 1] dx = \frac{-2}{k^2 + 1}. \quad (22)$$

The sum rule or k -integration gives of course

$$-2\pi N = \int_{-\infty}^{\infty} \rho_{\text{rel}}(k) dk = -2\pi \quad (23)$$

so that $N = 1$ as expected. There is more to say in this case, e.g. we find a zero-energy resonance or “half-bound” state which corresponds to the $k = 0$ limit, i.e.

$$\psi_{k=0} = -\tanh x. \quad (24)$$

As regards the delta potential we have

$$u(x) = g \delta(x) \quad (25)$$

where in principle we leave the *sign of g open*. As the support of the potential is a point, $\{0\}$, the solution for $x \neq 0$ is always asymptotic; there is also no odd wave.

For the spectral density we compute in this case

$$\rho_{\text{rel}}(k) = \int_{-\infty}^0 [|\psi_k(x)|^2 - 1] dx + \int_0^{\infty} [|\psi_k(x)|^2 - 1] dx \tag{26}$$

so that

$$\int_0^{\infty} [|\psi_k(x)|^2 - 1] dx = \int_0^{\infty} [|t(k)|^2 - 1] dx = -|b(k)|^2 L \tag{27}$$

for $L \rightarrow \infty$. This divergence is in fact *spurious* and canceled with the $x < 0$ contribution, i.e.

$$\int_{-\infty}^0 [|\psi_k(x)|^2 - 1] dx = \int_{-\infty}^0 [1 + |b(k)|^2 + 2\Re\{b(k) \exp(-2ikx)\} - 1] dx. \tag{28}$$

Now we define

$$2 \int_{-\infty}^0 \Re\{b(k) \exp(-2ikx)\} dx \equiv A + B \tag{29}$$

so that

$$A = 2 \int_{-\infty}^0 \Re\{b(k)\} \cos 2kx dx. \tag{30}$$

If we bear in mind that

$$2 \int_{-\infty}^0 \cos 2kx dx = \int_{-\infty}^{\infty} \cos 2kx dx = \int_{-\infty}^{\infty} \exp(2ikx) dx = 2\pi \delta(2k) = \pi \delta(k) \tag{31}$$

then we have that

$$A = \Re\{b(0)\} \pi \delta(k) \tag{32}$$

It is the case that

$$b(0) = -1 \tag{33}$$

for a generic potential, including the delta, because $\psi_{k=0}(x) = 0$ so

$$\exp(ikx) + b(k) \exp(-ikx) = 0 \text{ as } k \rightarrow 0 \tag{34}$$

and therefore $b(0) = -1$. The exception (*critical potentials*) occurs for a zero-energy resonance, see later, when $b(0) = 0$. On the other hand ($L \rightarrow +\infty$)

$$\frac{B}{\Im\{b(k)\}} = 2 \int_{-L}^0 \sin(-2kx) dx = \frac{1 - \cos 2kL}{k} \tag{35}$$

so that

$$B = \frac{\Im\{b(k)\}}{k} \tag{36}$$

as the oscillatory part $\cos 2kL$ gives no contribution *as a measure* when $L \rightarrow \infty$. Now for the delta potential itself we have (no odd wave)

$$f(k) = t(k) - 1 = b(k) = \frac{g}{2ik - g} \tag{37}$$

thus confirming $b(0) = -1$. So the relative spectral density as a whole is

$$\rho_{\text{rel}}(k) = -\pi \delta(k) + \frac{2g}{g^2 + 4k^2}. \tag{38}$$

Notice the delta piece, which will persist for any generic potential. Also, the dependence $\rho_{\text{rel}}(k) \propto 1/k^2$ for $k \gg 0$ is general as the phase shift itself will fall with $1/k$ (validity of the Born approximation) and we expect $\rho_{\text{rel}}(k) \propto \delta'(k)$. Now a k -integration would yield Levinson's theorem; taking care to isolate the sign(g) piece we get

$$-2\pi N = -\pi + \text{sign}(g) \pi. \tag{39}$$

In other words

$$N = \frac{1 - \text{sign}(g)}{2} \tag{40}$$

which is obviously correct: $N = 1$ (0) for $g < 0$ ($g > 0$).

5. GENERAL CALCULATION

Now we carry out the *general calculation*. Starting from

$$\psi''(x) + k^2\psi(x) = u(x)\psi(x) \tag{41}$$

we differentiate with respect to k (represented by the dot symbol), i.e. (Wellner, 1964; Kiers and van Dijk, 1996)

$$\dot{\psi}''(x) + 2k\dot{\psi} + k^2\dot{\psi}(x) = u(x)\dot{\psi}(x). \tag{42}$$

Why this unusual derivative? Because we expect the spectral density to depend on k -derivatives of the scattering amplitudes (or phases shift) as said before. Next we take real and imaginary parts of the wave function

$$\psi_k(x) = \Re\{\psi_k(x)\} + i\Im\{\psi_k(x)\} \tag{43}$$

since, the Schrödinger operator being real, each works separately. Now we construct the Wronskian for, first, $\Re\{\psi_k(x)\} := \Phi(x)$, which satisfies

$$\dot{\Phi}''(x) + 2k\dot{\Phi}(x) + k^2\dot{\Phi}(x) = u(x)\dot{\Phi}(x). \tag{44}$$

If we multiply and subtract in the usual way (e.g. to get the current) it is the case that

$$[\dot{\Phi}(x)\Phi'(x) - \Phi(x)\dot{\Phi}'(x)]' = 2k\dot{\Phi}^2(x) \tag{45}$$

or $(L \rightarrow \infty, \text{eventually})$

$$\Re\{I_k\} := \dot{\Phi}(x)\Phi'(x) - \Phi(x)\dot{\Phi}'(x)|_{-L}^L = 2k \int_{-L}^L \Phi^2(x) dx \tag{46}$$

which, together with the imaginary contribution, is the crucial result since it allows us to express the spectral density in terms of the asymptotic data. Next we define

$$A := \Re\{I_k\} \text{ at } L, \quad B := \Im\{I_k\}, \text{ at } L \tag{47}$$

$$C := \Re\{I_k\} \text{ at } -L, \quad D := \Im\{I_k\} \text{ at } -L \tag{48}$$

and

$$t(k) = |t(k)| \exp(i\varphi_t), \quad b(k) = |b(k)| \exp(i\varphi_r) \tag{49}$$

with $\Delta := \varphi_t + kL$. In doing so

$$A = |t(k)|^2 [k(\dot{\varphi}_t + L) + \cos \Delta \sin \Delta], \tag{50}$$

$$B = |t(k)|^2 [k(\dot{\varphi}_t + L) - \cos \Delta \sin \Delta]. \tag{51}$$

So the total forward contribution is

$$A + B = 2k |t(k)|^2 (\dot{\varphi}_t + L). \tag{52}$$

This is very nice: the factor $2k$ of (45) appears, as well as the derivative of the forward phase, (a hard datum; see below) while the L divergence will be spurious.

The calculation of the backward part is more involved as the wave function is

$$\psi(x \ll 0) = \exp(ikx) + b(k) \exp(-ikx). \tag{53}$$

By repeating the former method, now for the total backward contribution, we get (after some cancellations between C and D)

$$C + D = 4kL - 2|b(k)|^2 k(\dot{\varphi}_r - L) + 2b(k) \sin(\varphi_r + 2kL). \tag{54}$$

As regards the regularized spectral density $\rho_L(k)$ we have finally $A + B - C - D$ or

$$2k\rho_L(k) = 4kL + 2k\dot{\varphi}_t + 2k|b(k)|^2(\dot{\varphi}_r - \dot{\varphi}_t) + 2b(k) \sin(\varphi_r + 2kL) \tag{55}$$

So the final result is

$$-2\pi N = \int \left[\lim_{L \rightarrow \infty} (\rho_L(k) - \rho_L^{(0)}(k)) \right] dk \tag{56}$$

where as expected

$$\rho_L^{(0)}(k) = \int_{-L}^L dx = 2L \tag{57}$$

To sum up

$$-2\pi N = \int_{-\infty}^{\infty} \left[\dot{\varphi}_t + |b(k)|^2(\dot{\varphi}_r - \dot{\varphi}_t) + \frac{b(k)}{k} \sin(\varphi_r + 2kL) \right] dk. \quad (58)$$

As a matter of fact the first term of the integrand would be the *relative spectral density*. What about the back phase and the oscillatory third term? We find that, first,

$$\int_{-\infty}^{\infty} \frac{b(k)}{k} \sin(\varphi_r + 2kL) dk = \pi b(0) \quad (59)$$

as shown in Appendix. So the $\sin(\varphi_r + 2kL)/k$ integral is really $\pi\delta(k)$ as a distribution. This is completely rigorous because we are talking of measures (projection-valued measures) and the delta is itself a measure (not so the delta prime).

As regards the second term in (58) we write the *zurdo* contribution with $k > 0$ and then integrate from 0 to ∞ , i.e.

$$|b(k)|^2(\dot{\varphi}_r - \dot{\varphi}_t) + |\tilde{b}(k)|^2(\dot{\tilde{\varphi}}_r - \dot{\varphi}_t) = |b(k)|^2(\dot{\varphi}_r + \dot{\tilde{\varphi}}_r - 2\dot{\varphi}_t) = 0, \quad (60)$$

because (see (9) and (10)) $|b(k)| = |\tilde{b}(k)|$ and $2\varphi_t - \varphi_r - \tilde{\varphi}_r = \pi$. So the final expression for the relative spectral density is

$$\rho_{\text{rel}}(k) = \dot{\varphi}_t + \pi b(0)\delta(k), \quad k \geq 0 \quad (61)$$

see e.g. (Kiers and van Dijk, 1996; Sassoli de Bianchi, 1994). The first term, derivative of the forward phase, comes by no surprise and represents the germ of Levinson's theorem. The second one gives a universal contribution since

$$b(0) = -1 \text{ (generic); } b(0) = 0 \text{ (critical)} \quad (62)$$

as we already discussed. We remark here how the spectral density is a hard function and therefore the forward phase, but not the backward one, is a hard datum. Indeed, at least for finite-range potentials the forward amplitude can be expressed in terms of the bound states plus an integral over the modulus of the reflected amplitudes (Lamb, 1980). Again the usual proof is based on analytic continuation, whereas ours stems directly from the definition of hard data as spectral data.

We can easily compute the value of $\rho_{\text{rel}}(k)$ for large k . From the Born approximation

$$f(k) = t(k) - 1 \simeq \frac{1}{2ik} \int_{-\infty}^{\infty} u(x) dx \equiv \frac{-i}{2k} \langle u \rangle. \quad (63)$$

Hence

$$\tan \varphi_t(k) \simeq \frac{-\langle u \rangle}{2k} \simeq \varphi_t(k) \simeq \frac{-\langle u \rangle}{2k} \quad (64)$$

or

$$\rho_{\text{rel}}(k) = \frac{d\varphi_r(k)}{dk} \rightarrow \frac{\langle u \rangle}{2k^2} \quad (k \gg 0). \tag{65}$$

6. THE SUM RULE

The crowning result is Levinson’s theorem in one dimension: integrating (61) from $k = 0$ to $k = \infty$ we find

$$N = \frac{[\varphi_l(0) - \varphi_r(\infty)]}{\pi} - \frac{b(0)}{2} \tag{66}$$

as first given (except that he took only $b(0) = -1$) by Barton (1985).

We already showed that $b(0) = -1$ for a generic potential, that is, when the full wave function $\psi_k(x) \rightarrow 0$ for $k \rightarrow 0$. When the potential is critical the zero-energy wave function is non-zero, just starting from

$$\psi_{k \rightarrow 0} = [\exp(ikx) + b(k)\exp(-ikx)]_{k=0} = 1 \quad (x \ll 0) \tag{67}$$

and so $b(0) = 0$ for $u(x)$ critical. Hence then $|t(0)| = 1$, but the phase depends on the potential. In particular if $u(x)$ is even, the zero-energy resonance is either even or odd and therefore $t(0) = \pm 1$, $\psi_{k=0}(x)$ even/odd. For example, in the Pöschl-Teller case given by

$$u(x) = -\frac{\ell(\ell + 1)}{\cosh^2 x}, \quad \ell \text{ integer} \tag{68}$$

the zero-energy resonance has the parity of ℓ . Thus the first member $\ell = 1$ of the series obtains (see (24))

$$\psi_{k=0}(x) = -\tanh x. \tag{69}$$

Now we come to the most conspicuous aspect of the one-dimensional scattering, namely the one-half factor in the generic case (66). As hinted at by Barton, the reason is related to the fact that any attractive potential binds in one-dimension (this theorem seems due to Peierls, 1929). Then, the $u(x) = 0$ potential is critical, that is, increasing it a little bit in the attractive side produces a *bona fide* bound state. Indeed, $b(k) = 0$ for no potential, the earmark for a critical potential, namely transparency at $k = 0$. We like to call (68) supercritical because it is transparent, i.e. there is no reflection at any energy.

The phase of the forward amplitude is related to the determinant of the S -matrix. From (8) and the phase relation (10)

$$\begin{aligned} \text{Det}S(k) &= t\tilde{t} - b\tilde{b} = t^2 - |b|^2 \exp i(\varphi_r + \tilde{\varphi}_r) \\ &= |t|^2 \exp(2i\varphi_l) - e^{i\pi} |b|^2 \exp(2i\varphi_l) = \exp(2i\varphi_l) \end{aligned} \tag{70}$$

already noticed by many people, e.g. Jauch (1957).

Therefore, if $s(k) := \text{Det}S(k)$,

$$[N + b(0)/2] = \frac{-1}{2\pi i} \int \frac{\dot{s}}{s} dk \tag{71}$$

very similar to the conventional proof, as the integral can be performed in the complex plane Levinson (1949a).

The 1/2 contributions for critical potentials, both free $u = 0$ and interacting, are reminiscent of the η -invariant in the APS index theorem for manifolds with boundary; indeed, this can be seen explicitly in the supersymmetric formulation, in which it appears as the index of the Dirac operator, giving e.g. fermion numbers 1/2 (see Niemi and Semenoff, 1986; Fahri *et al.*, 1976). Moreover, the 1/2 value is characteristic of time-reversal invariant systems, both here and in the fractionization case.

7. FINAL REMARKS

In principle our method can be applied in arbitrary dimension D : the Schrödinger equation for scattering reads

$$\nabla^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) = u(\vec{r}) \psi(\vec{r}), \tag{72}$$

where $\vec{r} \in R^D$. Again $u(\vec{r})$ is real so taking real and imaginary parts and differentiating with respect to k , we can get e.g. for the real part $\Phi = \Re\{\psi\}$

$$\int_{\partial V} [\Phi \nabla \Phi - \Phi \nabla \Phi] d^{D-1} \sigma = 2k \int_V \Phi^2 d^D \vec{r}. \tag{73}$$

The first term is evaluated asymptotically in terms of

$$\psi_k(r) \rightarrow \exp(ikr \cos \theta) + r^{-(D-1)/2} f(\Omega) \exp(ikr) \tag{74}$$

as $r \gg 0$, without supposing $u(\vec{r}) = u(|\vec{r}|)$, i.e. non-central potentials are included. The second term in (73) is related, as before, to the spectral density. The calculation proceeds in the same way, except that for $D > 2$ the case $u(\vec{r}) = 0$ is not critical. We refrain to reproduce the well known results both for $D = 2$ and for $D > 2$ (for $D = 2$ see Newton, 1977, 1983, 1984, 1982; Cheney, 1984).

Non-local potentials require a different strategy, because then time reversal T does not necessarily hold. Here we want just to show how in one-dimension a non-local *real* potential, which is T invariant, gives, in general, the same result as the local case. The Schrödinger equation is now

$$\psi''(x) + k^2 \psi(x) = \int_{-\infty}^{\infty} u(x, y) \psi(y) dy, \tag{75}$$

where

$$u(x, y) = u(y, x)^* \tag{76}$$

in all cases from hermiticity of the Hamiltonian H . If moreover T reversal holds, $u(x, y)$ is real (hence symmetric) and we get from (72)

$$\dot{\psi}''(x) + k^2 \dot{\psi}(x) + 2k \psi = \int_{-\infty}^{\infty} u(x, y) \dot{\psi}(y) dy. \quad (77)$$

Once again it suffices to take real and imaginary parts and write the Wronskian to eliminate the term of the potential, so Levinson's theorem seems to hold untouched. However, for a non-local potential there might be exceptionally bound states embedded in the continuum; for the form of the theorem in these cases see (Martin, 1958).

As a final comment we want to compare the optical theorem in D dimensions (Boya and Murray, 1994) with this Levinson's theorem. Both depend on hard data, hence the appearance of the forward amplitude is to be expected. Also, they are interference-type formulae, linear in t , and represent the same completeness. For the optical theorem it is in coordinate space and takes the form of a conserved current, indeed the Noether current associated to the classical Lagrangian reproducing the time-dependent Schrödinger equation with a global phase invariance. For the spectral density the completeness appears in k -space; the sum rule for this case is a kind of global invariant of the problem.

The generalization of this presentation for the arbitrary D dimensional case is in progress.³

APPENDIX

Equation (59) is

$$I \equiv \int_{-\infty}^{\infty} \frac{b(k)}{k} \sin(\varphi_r + 2kL) dk \quad \text{for } L \rightarrow \infty. \quad (\text{A.1})$$

Define $2kL = k'$; then

$$\begin{aligned} I &= \int_{-\infty}^{\infty} b(k'/2L) \sin[\varphi_r(k'/2L) + k'] dk'/k' \\ &= b(0) \int_{-\infty}^{\infty} \sin(k') dk'/k' = \pi b(0) \end{aligned} \quad (\text{A.2})$$

as both $\varphi_r(0)$, $\tilde{\varphi}_r(0)$ and $b(0)$ are regular.

³ In collaboration with M. Aguado.

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